Combinatorial Sets of Reals, II

Spectra and Definability

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We will consider various extremal sets of reals, like

- maximal families of eventually different reals,
- maximal cofinitary groups,
- maximal independent families

and two specific aspects of their study:

- possible cardinalities;
- definability properties.

Maximal Eventually Different Families

Definition

A family $\mathscr{E} \subseteq {}^{\omega}\omega$ is eventually different (abbreviated e.d.) if for any two distinct $f, g \in \mathscr{E}$ there is $n \in \mathbb{N}$ such that

$$\forall m > n(f(m) \neq g(m)).$$

We write $f \neq^* g$. An e.d. family is maximal if it is not properly contained in any other e.d. family.

We denote such maximal families MED, their minimal cardinality $\mathfrak{a}_e$. For $f, g \in {}^{\omega}\omega$ if it is not the case that f, g are e.d., we write $f =^{\infty} g$.

Maximal cofinitary groups

Definition

- A group $\mathscr{G} \leq S_\infty$ is cofinitary if its elements are pairwise eventually different.
- A cofinitary group is maximal if it is not properly contained in any other cofinitary group.
- We denote such groups with MCG and their minimal cardinality $\mathfrak{a}_g$.

It is clear that MED and MCG are close relatives to maximal almost disjoint families and so $\mathfrak{a}_g$, $\mathfrak{a}_e$ are close relatives of $\mathfrak{a}$, the minimal cardinality of an infinite maximal almost disjoint subfamily of $[\omega]^\omega$.

To what extent are those distinct?

Let $\mathscr{M}$ denote the σ-ideal of meager sets and $\mathrm{non}(\mathscr{M})$ the minimal cardinality of a non-meager set.

- $\mathrm{non}(\mathscr{M})$ and $\mathfrak{a}$ are independent, while
- $\mathrm{non}(\mathscr{M}) \leq \mathfrak{a}_g, \mathfrak{a}_e$.

Comparing those combinatorial notions with respect to their projective complexity provides other clear distinctions:

- (A. Mathias) There are no analytic MAD families.
- (H. Horowitz, S. Shelah) There are Borel MED and Borel MCG.

One real at a time: Diagonalization

We can adjoin (via forcing) new desired reals one at a time and so recursively generate a MAD, MED, MCG.

- (Solovay) Almost disjoint coding.
- (Y. Zhang) A new generator for a cof. group.

Eliminating intruders

The ccc posets which naturally occur, apart from adjoining new elements to a given family, all have a second crucial property, which guarantees maximality at uncountable stages of uncountable cofinality in finite support iterations!

- Diagonalization allows us to obtain any uncountable size, as long as it is not of countable cofinality!
- What about $\aleph_\omega$?

Can we do better?

- (S. Hechler) We can adjoin a MAD family of arbitrary size with finite conditions, including families of cardinality $\aleph_\omega$, which eventually produced a model of $\mathfrak{a} = \aleph_\omega$ (J. Brendle, 2003).
- (F., A. Törnquist, 2015) We can also adjoin a MCG of arbitrary cardinality with finite conditions, including such max. groups of cardinality $\aleph_\omega$ and eventually obtain the consistency of $\mathfrak{a}_g = \aleph_\omega$.

Remark

The spectrum $\mathfrak{sp}(\mathfrak{a})$ is closed with respect to singular limits of countable cofinality. That is, if

$$\{\mu_i\}_{i\in\omega} \subseteq \mathfrak{sp}(\mathfrak{a})$$

is strictly increasing, then $\sup_{i\in\omega}\mu_i \in \mathfrak{sp}(\mathfrak{a})$.

Questions

The question, if either of

$$\mathfrak{sp}(\mathfrak{a}_e),\ \mathfrak{sp}(\mathfrak{a}_p) \text{ or } \mathfrak{sp}(\mathfrak{a}_g)$$

is closed with respect to singular limits is open!

MCG

- (Gao, Zhang) In L there is a MCG with a co-analytic generating set.
- (Kastermans) In L then there is a co-analytic MCG.
- (Horowitz, Shelah) There is a Borel MCG.

Question

What can we say about the existence of such nicely definable combinatorial sets of reals in models of large continuum?

Cohen forcing

Theorem (F., Schrittesser, Törnquist)

Assume $V = L$. Then there is a co-analytic MCG which is indestructible by Cohen forcing.

Corollary

The existence of a Π^1_1 MCG of cardinality $\aleph_1$ is consistent with $\mathfrak{c}$ begin arbitrarily large.

Our construction is inspired by the forcing method...

Definition: Coding a real into a group element

Let σ be a partial function from $\mathbb{N}$ to $\mathbb{N}$. Then

1. σ codes a finite string $t \in 2^l$ with parameter $m \in \mathbb{N}$ iff
$$(\forall k < l)\sigma^k(m) = t(k) \mod 2.$$
2. σ exactly codes $t \in 2^l$ with parameter m iff
$$\text{it codes } t \text{ and } \sigma^l(m) \text{ is undefined.}$$
3. σ codes $z \in 2^{\mathbb{N}}$ with parameter m iff
$$(\forall k \in \mathbb{N})\sigma^k(m) = z(k) \mod 2.$$

Outline

The group is recursively defined, in ω_1 steps, adding one generic permutation at a time, so that each new permutation codes a given real.

Definition: The partial order $\mathbb{Q}^z_{\mathscr{G}}$

Conditions of $\mathbb{Q}$ are triples $p = (s^p, F^p, \bar{m}^p)$ such that:

1. $(s^p, F^p) \in \mathbb{Q}_{\mathscr{G}}$, $\bar{m}^p$ is a partial function from F^p to $\mathbb{N}$
2. For any $w \in \mathrm{dom}(\bar{m}^p)$ there is $l \in \omega$ such that $w[s^p]$ exactly codes $z \restriction l$ with parameter $\bar{m}^p(w)$
3. $\dots$

with extension relation:

1. $(s^q, F^q, \bar{m}^q) \leq (s^p, F^p, \bar{m}^p)$ if and only if $(s^q, F^q) \leq_{\mathbb{Q}} (s^p, F^p)$ and $\bar{m}^q$ extends $\bar{m}^p$ as a function.

The generic group

Theorem

Let $\mathscr{G} \leq S_\infty$, $z \in 2^{\mathbb{N}}$, let G be $(M, \mathbb{Q}^z_{\mathscr{G}})$-generic filter and let

$$\sigma_G = \bigcup_{p \in G} s^p \in S_\infty.$$

1. Then $\langle \mathscr{G}, \sigma_G \rangle$ is cofinitary, isomorphic to $\mathscr{G} * \mathbb{F}(x)$.
2. If $\tau \in (S_\infty \backslash \mathscr{G}) \cap M$ is cofinitary, then $\langle \mathscr{G} \cup \{\sigma_G, \tau\} \rangle$ is not cofiniatry.
3. Any new permutation in $\langle \mathscr{G} \cup \{\sigma_G\} \rangle$ codes z.

To summarize

1. The existence of a co-analytic MCG of size $\aleph_1$ is consistent with

$$\mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}.$$

2. The existence of a co-analytic MED of size $\aleph_1$ is consistent with

$$\mathfrak{a}_e = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}.$$

How to obtain a model in which there is a co-analytic MED family of cardinality $\aleph_1$ and $\mathfrak{d} < \mathfrak{c}$?

Theorem (F., Schrittesser)

In the constructible universe L there is a co-analytic MED which remains maximal after countable support iterations or countable support products of Sacks forcing.

To summarize

The existence of a co-analytic MED family of cardinality $\aleph_1$ is consistent with

$$\mathfrak{a}_e = \mathfrak{d} = \aleph_1 < \mathfrak{c}.$$

Definition

A forcing notion $\mathbb{P}$ has the property ned iff for every countable $\mathscr{F}_0 \subseteq {}^\omega\omega$ and every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^\omega\omega$ such that

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is e.d. from } \check{\mathscr{F}}_0,$$

there are $h \in {}^\omega\omega$ which is e.d. from $\mathscr{F}_0$ and $p \in \mathbb{P}$ with

$$p \Vdash_{\mathbb{P}} \check{h} =^\infty \dot{f}.$$

Theorem

1. For ${}^\omega\omega$-bounding Suslin posets, the property ned is preserved under countable support iterations.
2. Sacks forcing, as well as its countable support products and iterations have property ned.

Theorem

Suppose $\mathscr{E}$ is a Σ^1_2 MED family. Then, there is a Π^1_1 MED family $\mathscr{E}'$ such that for any forcing $\mathbb{P}$, if $\mathscr{E}$ is $\mathbb{P}$-indestructible, then so is $\mathscr{E}'$.

1. (Törnquist) The existence of a Σ^1_2 definable MAD implies the existence of a Π^1_1 MAD.
2. (Brendle, F., Khomskii) The existence of a Σ^1_2 definable MIF implies the existence of a Π^1_1 MIF.
3. (F., Schilhan) The existence of a Σ^1_2 definable tower implies the existence of a Π^1_1tower.

However the question if the existence of a Σ^1_2 definable MCG implies the existence of a Π^1_1 one is still open.

Tightness

Observations

- If X is a set of functions, then $\bigcup X \subseteq \omega^2$.
- Similarly if $T \subseteq \omega^{<\omega}$ is a tree then $\bigcup T \subseteq \omega^2$.

Definition

Let X be a set of functions.

1. We say that X covers a tree T if $\bigcup T \subseteq \bigcup X$.
2. We say that X almost covers T if $\bigcup T \subseteq^* \bigcup X$.
3. If T is a tree and $t \in T$, then $T_t = \{s \in T : s \subseteq t \text{ or } t \subseteq s\}$.

The tree ideal generated by $\mathscr{E}$

Definition (F., C. Switzer)

1. The tree ideal generated by $\mathscr{E} \subseteq {}^\omega\omega$, denotes $\mathscr{I}_T(\mathscr{E})$, is the set of all trees $T \subseteq \omega^{<\omega}$ so that there are
$$t \in T \text{ and a finite } X \subseteq \mathscr{E}$$
so that
$$\bigcup T_t \subseteq^* \bigcup X.$$
2. A tree $T \subseteq \omega^{<\omega}$ is said to be in $\mathscr{I}_T(\mathscr{E})^+$ if for each $t \in T$ it is not the case that $\bigcup T_t$ can be almost covered by a finite $X \subseteq \mathscr{E}$.

Tight eventually different families

Definition

Let $T \subseteq \omega^{<\omega}$ be a tree, $g \in {}^\omega\omega$.

1. g densely diagonalizes T if for each $t \in T$ there is an $s \in T$ such that $t \subsetneq s$ and for some $k \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$ we have $s(k) = g(k)$.
2. That is, g densely diagonalizes T, if for every $t \in T$ there is a branch h through t in T such that $h =^\infty g$.

Definition

An eventually different family $\mathscr{E}$ is said to be tight if given any countable sequence $\{T_n\}_{n\in\omega} \subseteq \mathscr{I}_T(\mathscr{E})^+$ there is a single $g \in \mathscr{E}$ which densely diagonalizes all the T_n's.

Observations

- If $\mathscr{E}$ is a tight eventually different family, then it is maximal.
- MA(σ-linked) implies that every e.d. family $\mathscr{E}_0$, $|\mathscr{E}_0| < \mathfrak{c}$ is contained in a tight e.d. family.
- CH implies that tight eventually different families exist.

... and moreover

1. In the constructible universe L there is a co-analytic, Cohen indestructible tight e.d. family.
2. Thus (once again!) the existence of a co-analytic MED family is consistent with $\mathfrak{a}_e = \mathfrak{b} = \aleph_1 < \mathfrak{d} = \mathfrak{c}$.

Strong Preservation of Tightness

Definition: Strong preservation

Let $\mathbb{P}$ be a proper forcing notion and $\mathscr{E}$ a tight e.d. family. We say that $\mathbb{P}$ strongly preserves the tightness of $\mathscr{E}$ if for every sufficiently large θ and $M \prec H_\theta$ such that $p, \mathbb{P}, \mathscr{E}$ are elements of M,

if g strongly diagonalizes every elements of $M \cap \mathscr{I}_T(\mathscr{E})^+$,

then there is an $(M, \mathbb{P})$-generic $q \leq p$ such that q forces that

g densely diagonalizes every element of $M[\dot{G}] \cap \mathscr{I}_T(\mathscr{E})^+$.

Such a q is called an $(M, \mathbb{P}, \mathscr{E}, g)$-generic condition.

Theorem

Suppose $\mathscr{E}$ is a tight e.d. family. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a countable support iteration of proper forcing notions such that for all α,

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } \check{\mathscr{E}},$$

then $\mathbb{P}_\gamma$ strongly preserves the tightness of $\mathscr{E}$.

Observation

Thus, the notion of a tight eventually different family gives a uniform framework which applies to a long list of partial orders, including:

- Sacks,
- Miller rational perfect set forcing,
- Miller partition forcing,
- h-perfect trees
- Shelah's poset for diagonalizing a maximal ideal.

Theorem (F., Switzer)

The following inequalities are all consistent and in each case there is a tight eventually different family and a tight eventually different set of permutations of cardinality $\aleph_1$, respectively.

1. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p < \mathfrak{d} = \mathfrak{a}_T = 2^{\aleph_0}$
2. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} < \mathfrak{a}_T = 2^{\aleph_0}$
3. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} = \mathfrak{u} < non(\mathcal{N}) = cof(\mathcal{N}) = 2^{\aleph_0}$.
4. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = cof(\mathcal{N}) < \mathfrak{u}$.

Moreover, if we work over the constructible universe, we can provide co-analytic witnesses of cardinality $\aleph_1$ to each of

$$\mathfrak{a}, \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{i}, \mathfrak{u}$$

in the above inequalities.

Definition

We refer to a MCG $\mathscr{G}$ of cardinality μ as witnesses to

$$\mu \in \mathrm{sp}(\mathfrak{a}_g) = \{|\mathscr{G}| : \mathscr{G} \text{ is mcg}\}$$

and to values $\mu \in \mathrm{sp}(\mathfrak{a}_g)$ such that

$$\aleph_1 < \mu < \mathfrak{c}$$

as intermediate cardinalities (or values).

Definition: Good projective witnesses

A good projective witness to

$$\mu \in \mathrm{sp}(\mathfrak{a}_g)$$

is a MCG $\mathscr{G}$ of cardinality μ which is also of

lowest projective complexity,

i.e. there are no witnesses to μ whose definitional complexity lies strictly below that of $\mathscr{G}$ in terms of the projective hierarchy.

Question

What can we say about the definability properties of maximal cofinitary groups $\mathscr{G}$ such that

$$\aleph_1 < |\mathscr{G}| < \mathfrak{c}?$$

Observation

Note that a Σ^1_2 MCG must be either of size $\aleph_1$ or continuum (being the union of $\aleph_1$ many Borel sets). Therefore the lowest possible projective complexity of a witness to intermediate values in $\mathrm{sp}(\mathfrak{a}_g)$ is Π^1_2.

Theorem (F., Friedman, Schrittesser, Törnquist)

It is relatively consistent with ZFC that:

- $\mathfrak{c} \geq \aleph_3$ and
- there is a Π^1_2 MCG of size $\aleph_2$.

Thus, it is consistent that there is a Π^1_2 good projective witness to an intermediate value in $\mathrm{sp}(\mathfrak{a}_g)$.

Remark

The same holds for the spectrum of MED and MAD.

Theorem (F., Friedman, Schrittesser, Törnquist)

Let $2 \leq M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which

$$\mathfrak{a}_g = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and there is a Π^1_2 definable maximal cofinitary group fo size $\aleph_M$.

Remark

The analogous result holds for maximal families of eventually different reals, maximal families of eventually different permutations, maximal families of almost disjoint sets.

	$\aleph_1$	μ	$\mathfrak{c}$
	Π^1_1	Π^1_2	Borel
MED	✓	?	✓
MED	?	✓	✓
MCG	✓	?	✓
MCG	?	✓	✓

Question

Can we simultaneously have optimal projective witnesses for $\aleph_1$, $\mathfrak{c}$ and an intermediate value?

Independent Families

A family $\mathscr{A} \subseteq [\omega]^\omega$ is said to be independent for any two non-empty finite disjoint subfamilies $\mathscr{A}_0$ and $\mathscr{A}_1$ the set

$$\bigcap \mathscr{A}_0 \backslash \bigcup \mathscr{A}_1$$

is infinite. It is a maximal independent family if it is maximal under inclusion and

$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \text{is a m.i.f.}\}$$

Boolean combinations

For finite $h : \mathscr{A} \to \{0, 1\}$, we refer to $\mathscr{A}^h = \bigcap h^{-1}(0) \backslash \bigcup h^{-1}(1)$ as a boolean combination. If $h' \supseteq h$, we say that $\mathscr{A}^{h'}$ strengthen $\mathscr{A}^h$.

... and once again Maximality

Let $\mathscr{A}$ be an independent family.

- Note that, if $\mathscr{A}$ is maximal, then $\forall X \in [\omega]^\omega \backslash \mathscr{A} \exists h \in \mathrm{FF}(\mathscr{A})$ such that X does not split $\mathscr{A}^h$.
- If for each $X \in [\omega]^\omega \backslash \mathscr{A}$ and every $h \in \mathrm{FF}(\mathscr{A})$ there is a strengthening of $\mathscr{A}^h$ which is not split by X, we say that $\mathscr{A}$ is densely maximal.

Remark

The notion of dense maximality appears for the first time in the work of M. Goldstern and S. Shelah on the consistency of $\mathfrak{r} < \mathfrak{u}$.

Density filter

Let $\mathscr{A}$ be an independent family. The family of all $Y \subseteq \omega$ with the property that every $\mathscr{A}^h$ has a strengthening contained in Y is a filter, referred to as the the density filter and denoted $\mathrm{fil}(\mathscr{A})$.

Definition: Selective independence

A densely maximal independent family $\mathscr{A}$ is said to be selective if $\mathrm{fil}(\mathscr{A})$ is Ramsey.

Theorem (Shelah)

- Selective independent families exists under *CH*.
- They are indestructible by a countable support iterations and countable support products of Sacks forcing.

Remark

It is consistent that $\mathfrak{i} < \mathfrak{c}$. In fact the construction can be extracted from Shelah's proof of $\mathfrak{i} < \mathfrak{u}$.

Theorem (A. Miller)

There are no analytic maximal independent families.

Theorem (Brendle, F., Khomskii)

It is relatively consistent that $\mathfrak{i} = \aleph_1 < \mathfrak{c}$ with a co-analytic witness to $\mathfrak{i}$.

Recall that existence of a Σ^1_2 MIF implies the existence of a Π^1_1 MIF.

Optimal spectra?

	$\aleph_1$	μ	$\mathfrak{c}$	
MIF	✓	–	?	$V^{\mathbb{S}_\lambda} \vDash \mathrm{sp}(\mathfrak{i}) = \{\aleph_1, \mathfrak{c}\}$
MIF	–	–	✓	$V^{\mathbb{P}} \vDash \mathfrak{r} = \mathfrak{i} = \mathfrak{c}$

It is still open how to guarantee the existence of

- a good projective witnesses for two distinct cardinals in $\mathrm{sp}(\mathfrak{i})$, or
- a good projective witness for intermediate values.

Thank you for your attention!